$j - \omega$ ENSTROPHY CASCADES IN PHYSICAL SCALES OF 3D INCOMPRESSIBLE PLASMA

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ABSTRACT. Working directly from the 3D magnetohydrodynamical equations and entirely in physical scales a scenario is identified wherein the kinetic and magnetic enstrophies in a turbulent plasma exhibit a cascade over a range of scales the lower limit of which is a modified Kraichnan-type scale. Sufficient conditions include a hybrid geometric/smoothness assumption on the vorticity, a restraint on the scales over which the cascade transpires, and technical localization and modulation requirements on the evolution of the vorticity and current. Scale locality of the enstrophy flux is also examined.

1. INTRODUCTION.

In a series of recent publications, R. Dascaliuc and Z. Grujić established the existence of various turbulent cascades as well as the locality of the corresponding fluxes working in *physical scales* of incompressible fluid flows. The results rigorously follow directly from the Navier-Stokes equations (cf. [DaGr1], [DaGr2], [DaGr3], and [DaGr4]). This was achieved via a novel process of *ensemble averaging* which acts as a detector of significant *sign-fluctuations* of a physical density of interest. In the context of turbulent cascades, the *a priori* sign-changing density of interest is flux; if a quantity such as energy or enstrophy is being transferred from larger to smaller scales then the (suitably averaged) inward directed flux through shells of a given radius should be positive. That the setting is entirely physical provides a natural way to incorporate geometric information about the flow into the analytic framework. This was particularly useful in [DaGr3].

Because of the formal similarities between the two systems, mathematical results in the theory of NSE often illuminate and direct the study of MHD. For general remarks concerning the mathematical theory of MHD see [SeTe]. In this and a complementary paper, [BrGr1], the authors extend the framework of Dascaliuc and Grujić to study various cascades present in magnetohydrodynamic turbulence (see [DB] for the physical theory of MHD turbulence). The focus of [BrGr1] are turbulent events at the *u-b* level. There, the total energy flux is found to cascade over a range of scales the lower bound of which is a modified Taylor microscale. Further results are given for energies transported by specific mechanisms (advection by the magnetic field, for instance). Regarding the cascade for the fluid-advected kinetic energy plus fluid pressure flux, a role is identified for a numerically and phenomenologically motivated scale dependent dynamic alignment (cf. [B05, PeBo10, PoBh10, GS95]) which serves to partially deplete the non-linear coupling between the magnetic and velocity fields. In addition to studying turbulent cascades, [BrGr1] includes a scenario in which the predominant energy transfer is from the velocity to the magnetic field, results regarding locality of the fluxes, and a note detailing implications for *non-decaying* turbulence in flows governed by 3D NSE.

The present work identifies a set of physically reasonable assumptions which guarantee the existence of magnetic and kinetic enstrophy cascades across a range of physical scales as well as the

scale locality between scales of these cascades. The framework and trajectory of our efforts follow the work presented in [DaGr3] but, due to the numerous non-linear coupled terms evident in the current-vorticity formulation of 3D MHD, significant technical adaptations are needed. As in [DaGr3], a significant mathematical hurdle is to determine a range of physical scales over which the contribution of non flux-type terms to the ensemble averages can be interpolated between certain integral-scale quantities in order to ensure their detractions are small. In the incompressible fluid case, the only term requiring such treatment is the vortex stretching term and it is here that the geometry of coherent vortex structures plays an important role. The 3D MHD setting is considerably more complicated. The current-vorticity formulation includes a number of critical order advective terms which are structurally similar to the vortex stretching term but also includes a constructurally unique term, $\nabla u_l \times \nabla b_l$. Treatment of this last term necessitates several modifications to the assumptions prescribed in [DaGr3].

The most significant change between the assumptions in [DaGr3] and our efforts here is necessitated because the purely geometric criteria motivated by the existence of coherent vortex structures is not apparently helpful in the treatment of $\nabla u_l \times \nabla b_l$. Modified assumptions have been detailed in [Wu00] for ideal MHD and in [HeXin] in order to establish comparable regularity results as [CoFe93] and [BeBe] where the methods applied in [DaGr3] were pioneered. Our assumption is the hybrid geometric/smoothness condition of [HeXin] which, instead of asserting local coherence on the directions of vorticity, requires that, for a point x lying in a region of high kinetic turbulence – i.e. $|\nabla u|$ large – and for $|y|$ less than some critical scale, we have the inequality:

$$|\omega(x+y) - \omega(x)| < |\omega(x+y)||y|^{\frac{1}{2}}.$$

The above relationship, due to the fact that the strain tensor can be decomposed into singular integrals with kernels exhibiting certain cancellation properties over the unit sphere (cf. [Co94]), results in the same localized depletion of nonlinearities present in [DaGr3] for the vortex stretching term. It is worth highlighting that the hybrid geometric/smoothness condition is imposed solely on the vorticity field.

The differences between the fluid and plasma cases are also evidenced by altered dependencies of the Kraichnan-type scale. From our treatment of $\nabla u_l \times \nabla b_l$ we accumulate a dependence on an integral scale and energy-level space-time averaged quantity in addition to one at the enstrophy-level (the latter of these is the only one present in the fluid case). This is necessary to accommodate the *local* discrepancy between j and ∇b. That is, although $||j||_2 = ||\nabla b||_2$, the same cannot be said for $||\phi\, j||_2$ and $||\phi\, \nabla b||_2$ where ϕ is a smooth cut-off function having bounded support.

This paper is organized as follows. In Section 2 we recall the theoretical apparatus by which we study turbulence cascades in physical scales. In Section 3 we contextualize the general discussion of Section 2 to the case of 3D MHD and present the main theorem, Theorem 5. Section 4 is concerned with establishing necessary bounds for the proof of Theorem 5 while Section 5 contains the actual proof. Finally, in Section 6, we include corollary results establishing the scale locality of the enstrophy fluxes.

2. $(K_1, K_2)-$COVERS AND ENSEMBLE AVERAGES.

The main purpose of this section is to describe how *ensemble averaging* with respect to (K_1, K_2)-*covers* of an integral domain $B(0, R_0)$ can be used to establish *essential positivity* of an *a priori* sign-varying density over a range of physical scales associated with the integral domain. The application to turbulence is establishing the positivity of certain inward directed flux densities – i.e. the cascade is uni-directional from larger to smaller scales – as well as the near-constancy of the

averaged densities – i.e. the space-time averages over cover elements are all mutually comparable – across a range of scales.

The ensemble averages will be taken over collections of spatio-temporal averages of physical densities localized to cover elements of a particular type of covering – a so called (K_1, K_2)-*cover* – where the cover is over the region of turbulent activity. For simplicity, this region will be taken as a ball of radius R_0 centered at the origin and is henceforth referred to as the *integral domain*. The (K_1, K_2)-covers are now defined.

Definition 1. *Let* $K_1, K_2 \in \mathbb{N}$ *and* $0 \leq R \leq R_0$. *The cover of the integral domain* $B(0, R_0)$ *by the* n *balls,* $\{B(x_i, R)\}_{i=1}^n$ *is a* (K_1, K_2)-*cover at scale* R *if,*

$$\left(\frac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\frac{R_0}{R}\right)^3,$$

and, for any $x \in B(0, R_0)$, *x is contained in at most* K_2 *balls from the cover.*

In the hereafter all covers are understood to be (K_1, K_2)- covers at scale R. The positive integers K_1 and K_2 represent the maximum allowed *global* and *local multiplicities*, respectively.

In order to localize a physical density to a cover element we incorporate certain *refined* cut-off functions. For the cover element around the point x_i, let $\phi_i(x, t) = \eta(t)\psi(x)$ where $\eta \in C^\infty(0, T)$ and $\psi \in C_0^\infty(B(x_i, 2R))$ satisfy,

$$(1) \qquad 0 \leq \eta \leq 1, \qquad \eta = 0 \text{ on } (0, T/3), \qquad \eta = 1 \text{ on } (2T/3, T), \qquad \frac{|\partial_t \eta|}{\eta^\delta} \leq \frac{C_0}{T},$$

and,

$$(2) \qquad 0 \leq \psi \leq 1, \qquad \psi = 1 \text{ on } B(x_i, R), \qquad \frac{|\partial_i \psi|}{\psi^\rho} \leq \frac{C_0}{R}, \qquad \frac{|\partial_i \partial_j \psi|}{\psi^{2\rho-1}} \leq \frac{C_0}{R^2},$$

where $3/4 < \delta, \rho < 1$.

By ϕ_0 we denote the cut-off function associated with the integral domain – the ball centred at $x = 0$ of radius R_0 – satisfying the above properties.

Comparisons will be necessary between averaged quantities localized to cover elements at some scale R and averaged quantities at the integral scale. To accommodate this we impose several additional conditions for x_i near the boundary of $B(0, R_0)$. If $B(x_i, R) \subset B(0, R_0)$ we assume $\psi \leq \psi_0$. Alternatively, when $B(x_i, R) \not\subset B(0, R_0)$ we stipulate that $\psi = 1$ on $B(x_0, R) \cap B(0, R_0)$, satisfies (2), and we additionally have,

> $\psi = \psi_0$ on the intersection of $S(x_0, R_0, 2R_0)$ and the cone with apex at the origin and with boundaries passing through the intersection of the circle centered at the origin of radius R_0 and the boundary of $B(x_i, R)$,

and,

> $\psi = 0$ on the intersection of the three sets $B(0, R_0) \setminus B(x_i, 2R)$, $S(0, R_0, 2R_0)$, and the outside of the cone with apex at the origin and boundaries passing through the intersection of the circle centered at the origin of radius R_0 and the boundary of $B(x_i, 2R)$.

The above apparatus is employed to study properties of a physical density at a *physical scale R* associated with the integral domain $B(0, R_0)$ in a manner which we now illustrate. Let θ be a

physical density (e.g. a flux density) and define its localized spatio-temporal average on a cover element at scale R around x_i as,

$$\tilde{\theta}_{x_i,R} = \frac{1}{T} \int_0^T \frac{1}{R^3} \int_{B(x_i,2R)} \theta(x,t)\phi_i^\delta(x,t) \; dx \; dt,$$

where $0 < \delta \leq 1$, and let $\langle\Theta\rangle_R$ denote the ensemble average over localized averages associated with cover elements,

$$\langle\Theta\rangle_R = \frac{1}{n} \sum_{i=1}^n \tilde{\theta}_{x_i,R}.$$

Examining the values obtained by ensemble averaging the averages associated to a variety of covers at a fixed scale allows us to draw conclusions about the flux density θ at comparable and greater scales. For instance, stability (i.e. near constancy) of $\{\langle\Theta\rangle_R\}$ indicates that the sign of θ is essentially uniform at scales comparable to or greater than R. On the other hand, if the sign were not essentially uniform at scale R, particular covers could be arranged to enhance negative and positive regions and thus give a wide range of sign varying values in $\{\langle\Theta\rangle_R\}$. In order, then, to show the essential positivity of an *a priori* sign varying density θ at a scale R it is sufficient to show the positivity of $\{\langle\Theta\rangle_R\}$.

Conversely, if θ is an *a priori* non-negative density, then ensemble averages are all comparable to the integral scale average across the range $0 < R \leq R_0$. Precisely put, there exists K_* depending only on K_1 and K_2 so that,

$$(3) \qquad\qquad \frac{1}{K_*}\Theta_0 \leq \langle\Theta\rangle_R \leq K_*\Theta_0,$$

where,

$$\Theta_0 = \frac{1}{T} \int_0^T \frac{1}{R^3} \int_{B(0,2R_0)} \theta(x,t)\phi_0^\delta(x,t) \; dx \; dt.$$

The inequalities (3) follow in particular from the selection of our cut-off functions as well as the defining properties of our cover. Indeed, by directly comparing the integrands of the involved localized spatio-temporal averages to the integral domain spatio-temporal average, one sees that that,

$$(4) \qquad\qquad \frac{1}{K_1}\Theta_0 \leq \langle\Theta\rangle_R \leq K_2\Theta_0.$$

For additional discussion of (K_1, K_2)-covers and ensemble averages, including some computational illustrations of the process, see [DaGr3].

3. ENSTROPHY CASCADES.

Our mathematical setting is that of weak solutions to the magnetohydrodynamic equations (cf. [SeTe] for the essential theory). Define $\mathcal{V} = \{f \in L^2(\mathbb{R}^3) : \nabla \cdot f = 0 \text{ in the sense of distributions}\}$ and let V be the closure of $\mathcal{V}$ under the norm of the Sobolev space $(H^1(\mathbb{R}^3))^3$. By a solution to 3D MHD we mean a weak (distributional) solution to the following coupled system (3D MHD),

$$u_t - \triangle u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla P = 0,$$
$$b_t - \triangle b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0,$$
$$\nabla \cdot u = \nabla \cdot b = 0,$$
$$u(x,0) = u_0(x) \in V,$$
$$b(x,0) = b_0(x) \in V,$$

where the magnetic diffusivity and kinetic viscosity have been set to one and P is the total pressure.

Taking the curl of the above equations yields the following evolution equations for the vorticity and current, denoted ω and j respectively,

$$(5) \qquad \partial_t \omega - \Delta \omega = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - (b \cdot \nabla)j + (j \cdot \nabla)b,$$

$$(6) \qquad \partial_t j - \Delta j = (u \cdot \nabla)j - (j \cdot \nabla)u - (b \cdot \nabla)\omega + (\omega \cdot \nabla)b + 2\sum_{l=1}^{3} \nabla b_l \times \nabla u_l.$$

In our study we substitute for the inward kinetic and magnetic enstrophy fluxes through the boundary of a ball, $B = B(x_0, 2R)$,

$$-\int_{\partial B} \frac{1}{2}|\omega|^2 (u \cdot n)\, d\sigma = -\int_B (u \cdot \nabla)\omega \cdot \omega\, dx,$$

$$-\int_{\partial B} \frac{1}{2}|j|^2 (u \cdot n)\, d\sigma = -\int_B (u \cdot \nabla)j \cdot j\, dx,$$

the inward kinetic and magnetic enstrophy flux through a shell, $S(x_0, R, 2R)$, by incorporating a (nearly radial) cut-off function, ϕ. This cut-off function was defined in Section 2 and is chosen so that the gradient is directed inward. After multiplying $(u \cdot \nabla)\, \omega$ and $(u \cdot \nabla)\, j$ respectively by $\phi\, u$ and $\phi\, b$, we have the following realization of the *local kinetic and magnetic enstrophy fluxes at scale R around the point x_0*,

$$\Phi_{x_0, R}^{\omega} := \int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi)\, dx = -\int (u \cdot \nabla)\omega \cdot (\phi\omega)\, dx,$$

$$\Phi_{x_0, R}^{j} := \int \frac{1}{2}|j|^2 (u \cdot \nabla\phi)\, dx = -\int (u \cdot \nabla)j \cdot (\phi j)\, dx.$$

Formulas for the localized enstrophy fluxes are realized via the non-linear terms $(u \cdot \nabla)\, \omega$ and $(u \cdot \nabla)\, j$ by multiplying (5) and (6) respectively by $\phi\, \omega$ and $\phi\, j$ and integrating. In this manner we see that the localized kinetic enstrophy flux is given by,

$$
\begin{aligned}
F^{\omega}(t) := \int_0^t \int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi)\, dx\, ds &= \int \frac{1}{2}|\omega(x,t)|^2 \psi(x)\, dx + \int_0^t \int |\nabla\omega|^2 \phi\, dx\, ds \\
&\quad - \int_0^t \int \frac{1}{2}|\omega|(\partial_s\phi + \Delta\phi)\, dx\, ds - \int_0^t \int (\omega \cdot \nabla)u \cdot \phi\omega\, dx\, ds \\
&\quad + \int_0^t \int (b \cdot \nabla)j \cdot (\phi\omega)\, dx\, ds - \int_0^t \int (j \cdot \nabla)b \cdot (\phi\omega)\, dx\, ds \\
&= \int \frac{1}{2}|\omega(x,t)|^2 \psi(x)\, dx + \int_0^t \int |\nabla\omega|^2 \phi\, dx\, ds \\
&\quad + H^{\omega} + N_1^{\omega} + L^{\omega} + N_2^{\omega},
\end{aligned}
$$

(7)

while the localized magnetic enstrophy flux is given by,

$$F^j(t) := \int_0^t \int \frac{1}{2}|j|^2 (u \cdot \nabla \phi)\, dx\, ds = \int \frac{1}{2}|j(x,t)|^2 \psi(x)\, dx + \int_0^t \int |\nabla j|^2 \phi\, dx\, ds$$

$$- \int_0^t \int \frac{1}{2}|j|(\partial_s \phi + \Delta \phi)\, dx\, ds - \int_0^t \int (\omega \cdot \nabla)b \cdot \phi j\, dx\, ds$$

$$+ \int_0^t \int (b \cdot \nabla)j \cdot (\phi j)\, dx\, ds - \int_0^t \int (j \cdot \nabla)u \cdot (\phi j)\, dx\, ds$$

$$- \int_0^t \int \left(\sum_{l=1}^3 \nabla u_l \times \nabla b_l \right) \cdot (\phi j)\, dx\, ds$$

(8)
$$= \int \frac{1}{2}|j(x,t)|^2 \psi(x)\, dx + \int_0^t \int |\nabla j|^2 \phi\, dx\, ds$$

$$+ H^j + N_1^j + L^j + N_2^j + X.$$

To establish the existence of the enstrophy cascades we will show that the ensemble averages of localized spatio-temporal averages of the above densities associated to an arbitrary (K_1, K_2)-cover are positive and nearly constant across a range of scales. To achieve this several assumptions are in order. Before proceeding, we introduce several quantities. Fix a value K_* so that,

$$K_* \geq \max\{(K_1 K_2)^{1/2}, 3K_2/4, K_1\},$$

and set,

$$\alpha = 4K_P K_*^2,$$

where K_P is a constant which will be quantified in Section 5.

Our assumptions are as follows.

(A1) **Hybrid Geometric/Smoothness Assumption.** It is assumed for some threshold $M > 0$ that we have,

$$|\omega(x+y,t) - \omega(x,t)| \leq |\omega(x+y,t)||y|^{\frac{1}{2}},$$

provided $|y| < 2R_0 + R_0^{\frac{2}{3}}$, x in $\{|\nabla u| > M\}$, and $\omega(x+y) \neq 0$.

Remark 2. This assumption is less satisfying than its analogue in the fluid case of [DaGr3] where it was sufficient to assume the numerically and observationally motivated assumption of *coherence of the direction of vorticity*. While there is current to current and vorticity to vorticity alignment in 3D MHD, such a geometric consideration does not appear to sufficiently deplete the numerous nonlinearities (the fluid approach breaks down when examining the term $\nabla u_l \times \nabla b_l$), which explains our need for a different assumption.

(A2) **Modified Kraichnan-Type Scale.** Let e_0, E_0, and P_0 denote the time averaged total energy, total enstrophy, and modified total palenstrophy at the integral scale. Precisely,

$$e_0 = \frac{1}{T} \int_0^T \frac{1}{R_0^3} \int \phi_0^{4\rho-3} \left(\frac{|u|^2}{2} + \frac{|b|^2}{2} \right) dx\, ds,$$

$$E_0 = \frac{1}{T} \int_0^T \frac{1}{R_0^3} \int \phi_0^{2\rho-1} \left(|\omega|^2 + |j|^2 \right) dx\, ds,$$

$$P_0 = \frac{1}{T} \int_0^T \frac{1}{R_0^3} \int \phi_0 \left(|\nabla \omega|^2 + |\nabla j|^2 \right) dx\, ds + \frac{1}{TR_0^3} \int \frac{1}{2}|\omega(x,T)|^2 \psi_0(x)\, dx.$$

The modification of palenstrophy is due to the nature of the temporal cut-off; in addition, note that the cut-off's are modified for technical reasons.

Set,

$$\mathcal{E}_0 = \left(\frac{E_0}{P_0}\right)^{\frac{1}{2}},$$

and,

$$\varepsilon_0 = \left(\frac{e_0}{P_0}\right)^{\frac{1}{4}}.$$

Define the modified Kraichnan-type scale σ_0 by,

$$\sigma_0 = \max\{\mathcal{E}_0, \varepsilon_0\}.$$

Our assumption (A2) is that, for some constant $\beta = \beta(M, K_1, K_2, \int_0^T ||\omega||_2^2 \, dt)$ with $0 < \beta < 1$ (the precise value will be identified later), we have,

$$\sigma_0 < \beta R_0.$$

Remark 3. The Kraichnan-type scale determines the lower limit of scales at which the enstrophy cascade occurs. For us this is realized by restricting to scales R with $\sigma_0/\beta < R$. In comparison to the analogous and identically named parameter in the 3D NSE case we here see a correction of β by a power of $1/2$ necessitated by the emergence of energy-level quantities in Section 4.

(A3) **Localization and Modulation** – Because $\int_0^T ||\omega||_2^2 \, ds$ is an *a priori* bounded quantity (cf. [SeTe]), for a given constant $C_0 > 0$ there exists R_0^* so that for any $R_0 \leq R_0^*$ we have

$$\left(\int_0^T ||\omega||^2_{L^2(B(0,2R_0+R_0^{\frac{2}{3}}))} \, ds\right)^{1/2} \leq \frac{1}{C_0}.$$

The *localization assumption* on R_0, the radius of the integral scale, is that, for $C_0 = \alpha$, we have $R_0 \leq R_0^*$. A precise (up to certain parameters) value for α will materialize in the proof.

The *modulation assumption* imposes a restriction on the time evolution of the integral-scale kinetic and magnetic enstrophies across $(0, T)$ consistent with our choice of the temporal cut-off. Precisely,

$$\int |\omega(x,T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_s \int |\omega(x,s)|^2 \psi_0(x) \, dx,$$

$$\int |j(x,T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_s \int |j(x,s)|^2 \psi_0(x) \, dx.$$

Remark 4. Regarding localization, we have essentially introduced an upper bound on the range of scales across which the enstrophy cascade transpires; it is likely smaller than the scale at which the cascade initiates. This restriction is technical; a lemma to be included in a future paper will show that the near-constancy across a bounded range of scales extends to a range beyond that bound. A speculative statement of this lemma can be found in [DaGr3].

Using the terminologies of the above assumptions we are ready to state our main result which establishes the positivity and near-constancy of the kinetic and magnetic enstrophy fluxes across a range of physical scales.

Theorem 5. *If a weak solution u, b of 3D MHD satisfies (A1)-(A3) on $B(0, 2R_0) \times (0, T)$ with initial data in V then,*

$$\frac{1}{4K_*} P_0 \le \langle \Phi_\omega \rangle_R, \langle \Phi_j \rangle_R \le 4K_* P_0,$$

for all $\frac{\sigma_0}{\beta} \le R \le R_0$ and K_* dependent only on K_1 and K_2.

Remark 6. It will be plain that the *localization estimates* to be presented in the following section imply that (A1) alone guarantees smoothness over the spatio-temporal integral domain; hence, we are effectively concerned with the global-in-space ($\mathbb{R}^3$) weak solutions that are smooth over the integral domain. However, since we do not impose *any boundary conditions* on the integral domain, the control over the 'smooth' norms is *strictly local*.

Before continuing to the proof of Theorem 5 we observe that enstrophy flux locality – kinetic to kinetic, kinetic to magnetic, and magnetic to magnetic enstrophy exchanges are predominantly between scales of comparable size – is an immediate corollary. Discussion of this corollary and its precise statement is withheld until Section 6.

To prove Theorem 5, we will confine the localized (to a ball of radius R centered at x_0) kinetic/magnetic enstrophy flux between scale- and cover-independent multiples of the localized (to the integral domain) total-palenstrophy, P_0. In particular, for an appropriate fixed quantity K_* we show, for any cover element centered at x_0 of scale R, that the space-time averaged inward fluxes of the magnetic and kinetic enstrophies are both nearly constant, ie,

$$\frac{1}{4K_*} P_0 \le \frac{1}{T} \int_0^T \frac{1}{R^3} \Phi_{x_0,R}^\omega \, ds, \ \frac{1}{T} \int_0^T \frac{1}{R^3} \Phi_{x_0,R}^j \, ds \le 4K_* P_0.$$

In order to achieve this certain bounds are needed. This is the subject of Section 4. The proof of Theorem 5 is presented in Section 5.

4. BOUNDS.

In this section each of the terms from (7) and (8) are bounded by quantities which can be related to e_0, E_0, and P_0 via the apparatus of ensemble averaging with respect to (K_1, K_2)-covers at scale R. Throughout, we limit our consideration to a fixed ball of radius R and suppress the corresponding subscripts. We label various scalars by K, K_e, K_E, and K_P and note these are dependent on K_1, K_2 and quantities determined by structural properties of 3D MHD.

Bounds for the linear terms in (7) and (8) follow simply from properties of the spatial cut-off (see (2)):

$$H := H^\omega + H^j \le \frac{K_E}{R^2} \int_0^T \int \phi^{2\rho-1} \big(|\omega|^2 + |j|^2 \big) \, dx \, ds.$$

Before proceeding to bound the nonlinear terms a digression is necessary to introduce the kinematic framework derived in [Co94] and adapted to MHD in [Wu00] and [HeXin]. Recall that the deformation tensor of the velocity field can be decomposed as the sum of a symmetric component, the strain tensor of u, S, and a skew component, $j \times \cdot$. Put precisely,

(9) $$\nabla u = S + \frac{1}{2} \omega \times .$$

The operators in the above decomposition have the following singular integral representations:

(10) $$\omega(x) = \frac{1}{4\pi} \, P.V. \int \sigma(\hat{y}) \omega(x + y) \frac{dy}{|y|^3},$$

and,

$$(11) \qquad S(x) = \frac{3}{4\pi} \, P.V. \int M(\hat{y}, \omega(x+y)) \frac{dy}{|y|^3},$$

for,

$$\hat{y} = \frac{y}{|y|}, \sigma(\hat{y}) = 3\hat{y} \otimes \hat{y} - I, \text{ and } M(\hat{y}, f) = \frac{1}{2} \left(\hat{y} \otimes (\hat{y} \times f) + (\hat{y} \times f) \otimes \hat{y} \right).$$

A key feature for our treatment of the term ∇u_l will follow from the fact that σ and M (the latter when f is held constant as a function of y) have mean zero on the unit sphere. Integral operators such as these are the topic [St93] and [St70]. We connect the above to the term $\nabla u_l \times \nabla b_l$ by noting for the unit vector e_l we have,

$$\nabla u_l = \nabla u \, e_l + \frac{1}{2}\omega \times e_l.$$

Using the zero mean value property we write,

$$|\nabla u_l| \leq \left| P.V. \int_{|y|<R^{2/3}} \left(\frac{1}{4\pi}\sigma(\hat{y})\big(\omega(x+y) - \omega(x)\big) + \frac{3}{4\pi}M(\hat{y}, \omega(x+y) - \omega(x)) \right) \frac{dy}{|y|^3} \right|$$
$$+ \left| \int_{|y|\geq R^{2/3}} \left(\frac{1}{4\pi}\sigma(\hat{y})\omega(x+y) + \frac{3}{4\pi}M(\hat{y}, \omega(x+y)) \right) \frac{dy}{|y|^3} \right|$$
$$= I_1 + I_2.$$

Our treatment is now divided between I_1 and I_2. For the former, the hybrid geometric-smoothness assumption (A1) entails that,

$$I_1 \leq K(\sigma, M) \int_{|y|\leq R^{2/3}} |\omega(x+y) - \omega(x)| \frac{dy}{|y|^3}$$
$$\leq K(\sigma, M) \int_{|y|\leq R^{2/3}} |\omega(x+y)| \frac{dy}{|y|^{5/2}},$$

and, therefore, by the Hardy-Littlewood-Sobolev inequality (cf. Chapter V of [St93]),

$$(12) \qquad ||I_1||_3 \leq K||\omega||_{L^2(B(x_i, R^{2/3}))}.$$

Regarding I_2, Hölder's inequality allows that,

$$I_2 \leq K(\sigma, M) \int_{|y|\geq R^{2/3}} \frac{1}{|y|^2} \frac{\omega(x+y)}{|y|} \, dy$$
$$\leq K(\sigma, M) \left(\int_{|y|\geq R^{2/3}} \frac{1}{|y|^4} \, dy \right)^{\frac{1}{2}} \left(\int_{|y|\geq R^{2/3}} \frac{|\omega(x+y)|^2}{|y|^2} \, dy \right)^{\frac{1}{2}}$$
$$(13) \qquad \leq K(\sigma, M) \frac{1}{R^{1/3+2/3}}||\omega||_{L^2(\mathbb{R}^3)} = \frac{K}{R}||\omega||_{L^2(\mathbb{R}^3)}.$$

Note that we can apply the exact same argument to ω alone to obtain,

$$(14) \qquad \omega(x) = \frac{1}{4\pi}P.V. \int_{|y|<R^{2/3}} \sigma(\hat{y})(\omega(x+y) - \omega(x))\frac{dy}{|y|^3} + \frac{1}{4\pi}\int_{|y|\geq R^{2/3}} \sigma(\hat{y})\omega(x+y)\frac{dy}{|y|^3}$$
$$\leq K||\omega||_{L^2(B(x_i, R^{2/3}))} + \frac{K}{R}||\omega||_{L^2(\mathbb{R}^3)}.$$

We are now ready to establish bounds on the coupled non-linear terms. We begin with the most involved, the term involving $\nabla u_l \times \nabla b_l$, that labelled X, as this will illustrate many of the computational steps necessary for the other terms. We will show that,

$$(15) \quad \int_0^T \int \phi j \cdot \nabla u_l \times \nabla b_l \, dx \, ds \leq \frac{K_P}{\alpha} \left(\frac{1}{2} \sup_s ||\psi^{\frac{1}{2}} j||_2^2 + \int_0^T \int \phi |\nabla j|^2 \, dx \, ds \right)$$
$$+ \frac{K_E}{R^2} \int_0^T \int \phi^{2\rho-1} |j|^2 \, dx \, ds + \frac{\alpha^2 K_e}{R^4} \int_0^T \int \phi^{4\rho-3} \frac{|b|^2}{2} \, dx \, ds.$$

We begin by splitting the spatial integral into the regions where $|\nabla u| \geq M$ and the complement. Considering the complement,

$$\int_0^T \int_{|\nabla u| \leq M} \phi j \cdot \nabla u_l \times \nabla b_l \, dx \, ds \leq \int_0^T \int_{|\nabla u| \leq M} M |\phi^{1/2} j| |\phi^{1/2} \nabla b_l| \, dx \, ds$$
$$\leq \frac{M^2}{R^2} \int_0^T \int \phi^{2\rho-1} |j|^2 \, ds + \frac{1}{R^2} \int_0^T \int \phi^{2\rho-1} |\nabla b_l|^2 \, ds.$$

The second integral above can be expressed in terms of energy and palenstrophy level terms. Indeed, for k, h, and l distinct elements of $\{1, 2, 3\}$,

$$\int_0^T \int \phi^{2\rho-1} (\partial_i b_l)^2 \, dx \, ds = -\int_0^T \int \phi^{2\rho-1} b_l \partial_i \partial_i b_l \, dx \, ds - \int_0^T \int \partial_i \phi^{2\rho-1} b_l \partial_i b_l \, dx \, ds$$
$$= \int_0^T \int \phi^{2\rho-1} b_l (\partial_k j_h - \partial_h j_k) \, dx \, ds + \frac{1}{2} \int_0^T \int \partial_i \partial_i \phi^{2\rho-1} b_l^2 \, dx \, ds$$
$$(16) \qquad \leq 2 \int_0^T \int \phi^{2\rho-1} |b| |\nabla j| \, dx \, ds + \frac{1}{2} \int_0^T \int |\partial_i \partial_i \phi^{2\rho-1}| |b|^2 \, dx \, ds$$
$$(17) \qquad \leq \int_0^T \left(2\alpha^{\frac{1}{2}} \phi^{2\rho-3/2} |b| \right) \left(\alpha^{-\frac{1}{2}} \phi^{1/2} |\nabla j| \right) dx \, ds$$
$$+ \frac{1}{R^2} \int_0^T \int \phi^{4\rho-3} |b|^2 \, dx \, ds.$$

Applying Young's inequality to the first term of (17) gives us a final bound for the case when $|\nabla u| < M$,

$$\int_0^T \int_{|\nabla u| \leq M} \phi j \cdot \nabla u_l \times \nabla b_l \, dx \, ds \leq \frac{1}{\alpha} \int_0^T \int \phi |\nabla j|^2 \, dx \, ds + \frac{K_E}{R^2} \int_0^T \int \phi^{2\rho-1} |j|^2 \, dx \, ds$$
$$+ \frac{\alpha K_e}{R^4} \int_0^T \int \phi^{4\rho-3} |b|^2 \, dx \, ds.$$
$$(18) \qquad \leq \frac{1}{\alpha} \left(\frac{1}{2} \sup_s ||\psi^{\frac{1}{2}} j||_2^2 + \int_0^T \int \phi |\nabla j|^2 \, dx \, ds \right)$$
$$+ \frac{K_E}{R^2} \int_0^T \int \phi^{2\rho-1} |j|^2 \, dx \, ds + \frac{\alpha^2 K_e}{R^4} \int_0^T \int \phi^{4\rho-3} \frac{|b|^2}{2} \, dx \, ds.$$

Looking now at the regions of high kinetic intensity – i.e. $|\nabla u| \geq M$ – we split into two cases using the decomposition for ∇u_l,

$$\left| \int_0^T \int_{|\nabla u| \geq M} \phi j \cdot \nabla u_l \times \nabla b_l \, dx \, ds \right| \leq \int_0^T \int_{|\nabla u| \geq M} I_1 |\phi^{1/2} \nabla b_l| |\phi^{1/2} j| \, dx \, ds$$
$$+ \int_0^T \int_{|\nabla u| \geq M} I_2 |\phi^{1/2} \nabla b_l| |\phi^{1/2} j| \, dx \, ds.$$

For the integral involving I_1, the hybrid geometric/smoothness assumption plus the localization and modulation assumptions will serve to minimize palenstrophy level terms. Applications of Hölder's inequality and the bound (12) and subsequently the Gagliardo-Nirenberg inequality and Young's inequality yields an initial bound,

(19)
$$\int_0^T \int_{|\nabla u| \geq M} I_1 |\phi^{1/2} \nabla b_l| |\phi^{1/2} j| \, dx \, ds \leq K \left(\int_0^T \|\omega\|_{L^2(B(x_i, R^{2/3}))}^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^T \|\phi^{\frac{1}{2}} \nabla b_l\|_6^2 \|\phi^{\frac{1}{2}} j\|_2^2 \, ds \right)^{\frac{1}{2}}$$
$$\leq \frac{1}{\alpha} \left(\frac{1}{2} \sup_s \|\psi^{\frac{1}{2}} j\|_2^2 + K \int_0^T \|\nabla(\phi^{\frac{1}{2}} \nabla b_l)\|_2^2 \, ds \right)$$
$$\leq \frac{1}{\alpha} \left(\frac{1}{2} \sup_s \|\psi^{\frac{1}{2}} j\|_2^2 + K \int_0^T \|\nabla \phi^{\frac{1}{2}} \otimes \nabla b_l\|_2^2 \, ds \right.$$
$$\left. + K \int_0^T \|\phi^{\frac{1}{2}} \nabla \nabla b_l\|_2^2 \, ds \right).$$

where the constant α emerges from the localization assumption, (A3). We further decompose the last two terms. For the first, by expanding the integral, using the bound (16), applying Young's inequality, and employing the properties of our cut-off functions, we have the following string of bounds:

$$\frac{K}{\alpha} \int_0^T \|\nabla \phi^{\frac{1}{2}} \otimes \nabla b_l\|_2^2 \, ds = \frac{K}{\alpha} \int_0^T \int (\partial_i \phi^{\frac{1}{2}})^2 (\partial_j b_l)^2 \, dx \, ds$$
$$\leq \frac{K}{\alpha R^2} \int_0^T \int \phi^{2\rho-1} (\partial_j b_l)^2 \, dx \, ds$$
$$\leq \frac{K}{\alpha R^2} \int_0^T \int \phi^{2\rho-1} |b| |\nabla j| \, dx \, ds + \frac{K}{\alpha R^2} \int_0^T \int |\partial_i \partial_i \phi^{2\rho-1}| |b|^2 \, dx \, ds$$
$$\leq \frac{1}{\alpha} \int_0^T \int \phi |\nabla j|^2 \, dx \, ds + \frac{K}{\alpha R^4} \int_0^T \int \phi^{4\rho-3} |b|^2 \, dx \, ds.$$

The last term of (19) can be expressed in a similar fashion as the above but with several additional steps. To begin, observe that by the product rule and integration by parts, we have the identity,

$$\|\phi^{\frac{1}{2}} \nabla \nabla b_l\|_2^2 = \int \phi (\partial_i \partial_j b_l)^2 \, dx = \int (\partial_i \partial_j \phi)(\partial_j b_l)(\partial_i b_l) \, dx + \int \phi (\partial_j \partial_j b_l)(\partial_i \partial_i b_l) \, dx$$
$$+ \int (\partial_j \phi)(\partial_j b_l)(\partial_i \partial_i b_l) \, dx + \int (\partial_i \phi)(\partial_j \partial_j b_l)(\partial_i b_l) \, dx.$$

We next bound each of the terms in the above identity. For the first, by Young's inequality, we have,

$$\frac{K}{\alpha} \int (\partial_i \partial_j \phi)(\partial_j b_l)(\partial_i b_l) \, dx \leq K \int |\partial_i \partial_j \phi||\partial_j b_l||\partial_i b_l| \, dx$$

$$\leq \frac{K}{R^2} \int \phi^{2\rho-1} |\partial_j b_l||\partial_i b_l| \, dx$$

$$= \frac{K}{R^2} \int \left(\phi^{\rho-1/2} |\partial_j b_l| \right) \left(\phi^{\rho-1/2} |\partial_i b_l| \right) \, dx$$

$$\leq \frac{K}{R^2} \int \phi^{2\rho-1} |\nabla b_l|^2 \, dx.$$

Moving on, since b is divergence free and assuming k, h, and l are distinct, we have,

$$\frac{K}{\alpha} \int (\partial_j \phi)(\partial_j b_l)(\partial_i \partial_i b_l) \, dx = \frac{K}{\alpha} \int (\partial_j \phi)(\partial_j b_l)\left(\partial_h j_k + \partial_k j_h\right) \, dx$$

$$\leq \frac{K}{\alpha} \int |\partial_j \phi||\partial_j b_l||\partial_h j_k| \, dx$$

$$\leq \int \left(\frac{K}{\alpha^{\frac{1}{2}} R} \phi^{\rho-1/2} |\partial_j b_l| \right) \left(\frac{1}{\alpha^{\frac{1}{2}}} \phi^{\frac{1}{2}} |\partial_h j_k| \right) \, dx$$

$$\leq \frac{K}{\alpha R^2} \int \phi^{2\rho-1} |\nabla b_l|^2 \, dx + \frac{1}{\alpha} \int \phi |\nabla j|^2 \, dx$$

$$\leq \frac{K}{R^4} \int_0^T \int \phi^{4\rho-3} |b|^2 \, dx \, ds + \frac{1}{\alpha} \int_0^T \int \phi |\nabla j|^2 \, dx \, ds.$$

Using Young's inequality and properties of the cut-off function, the remaining term is bounded as,

$$\frac{K}{\alpha} \int \phi (\partial_j \partial_j b_l)(\partial_i \partial_i b_l) \, dx \leq \frac{K}{\alpha} \int \phi (\partial_i \partial_i b_l)^2 \, dx + \frac{K}{\alpha} \int \phi (\partial_j \partial_j b_l)^2 \, dx$$

$$\leq \frac{K_P}{\alpha} \int \phi |\nabla j|^2 \, dx.$$

Combining the above (and observing $\alpha > 1$) gives a final bound for the term involving I_1:

$$\int_0^T \int_{|\nabla u| \geq M} \phi j \cdot \nabla u_l \times \nabla b_l \, dx \, ds \leq \frac{K_P}{\alpha} \left(\frac{1}{2} \sup_s ||\psi^{\frac{1}{2}} j||_2^2 + \int_0^T \int \phi |\nabla j|^2 \, dx \, ds \right)$$

(20)
$$+ \frac{\alpha^2 K_e}{\alpha R^4} \int_0^T \int \phi^{4\rho-3} |b|^2 \, dx \, ds$$

Turning now to the non-singular case of our decomposition of $|\nabla u_l|$, we again use α to de-emphasize non-localization-apt palenstrophy level quantities. Its emergence is forced upon an application of Young's inequality with a reciprocal cost to an energy level quantity. Using the direct estimate (13) on $|I_2|$ and then applying Hölder's inequality, Young's inequality, and the same

steps used to obtain and proceed from (17), we see that,

$$\int_0^T \int_{|\nabla u|\geq M} I_2 |\phi^{1/2}\nabla b_l||\phi^{1/2}j| \, dx \, ds \leq \frac{K}{R}\left(\int_0^T ||\omega||_2^2 \, ds\right)^{\frac{1}{2}}\left(\int_0^T ||\phi^{\frac{1}{2}}j||_2^2||\phi^{\frac{1}{2}}\nabla b_l||_2^2 \, ds\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{\sqrt{\alpha}}\sup_s ||\psi^{\frac{1}{2}}j||_2\right)\left(\frac{\alpha K}{R^2}\int_0^T ||\phi^{\frac{1}{2}}\nabla b_l||_2 \, ds\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\alpha}\frac{1}{2}\sup_s ||\psi^{\frac{1}{2}}j||_2^2 + \frac{\alpha K}{R^2}\int_0^T ||\phi^{\frac{1}{2}}\nabla b_l||_2 \, ds$$

(21)
$$\leq \frac{1}{\alpha}\left(\frac{1}{2}\sup_s ||\psi^{\frac{1}{2}}j||_2^2 + \int_0^T \int \phi|\nabla j|^2 \, dx \, ds\right)$$

$$+ \frac{\alpha^2 K_e}{R^4}\int_0^T \int \phi^{4\rho-3}\frac{|b|^2}{2} \, dx \, ds.$$

Combining the bounds (18), (20), and (21) establishes the bound (15) and concludes our discussion of the term involving $\nabla u_l \times \nabla b_l$.

Bounds for the remaining four critical order nonlinear terms, N_1^ω, N_2^ω, N_1^j, and N_2^j, are now attended. The processes for estimating N_1^ω and N_2^j are identical up to labeling. We only illustrate the latter. Splitting the space integral, when $|\nabla u| < M$, we have,

$$N_2^j = \int_0^T \int_{|\nabla u|<M} (j\cdot\nabla)u\cdot\phi j \, dx \, ds \leq M\int_0^T ||\phi^{\frac{1}{2}}j||_2^2 \, dx \, ds.$$

When $|\nabla u| \geq M$, noting by direct comparison that $|j| \leq \sqrt{5}|\nabla b|$, we bound the quantity,

$$\int_0^T \int_{|\nabla u|\geq M} |\nabla u||\phi^{\frac{1}{2}}\sqrt{5}\nabla b||\phi^{\frac{1}{2}}j| \, dx \, ds,$$

in an identical fashion to the unique-to-MHD term. The resultant final bounds are,

$$N_2^j = \int_0^T \int (j\cdot\nabla)u\cdot\phi j \, dx \, ds \leq \frac{K_P}{\alpha}\left(\frac{1}{2}\sup_s ||\psi^{\frac{1}{2}}j||_2^2 + \int_0^T \int \phi|\nabla j|^2 \, dx \, ds\right)$$

(22)
$$+ \frac{K_E}{R^2}\int_0^T \int \phi^{2\rho-1}|j|^2 \, dx \, ds + \frac{\alpha^2 K_e}{R^4}\int_0^T \int \phi^{4\rho-3}\frac{|b|^2}{2} \, dx \, ds,$$

and,

$$N_1^\omega = \int_0^T \int (\omega\cdot\nabla)u\cdot\phi\omega \, dx \, ds \leq \frac{K_P}{\alpha}\left(\frac{1}{2}\sup_s ||\psi^{\frac{1}{2}}\omega||_2^2 + \int_0^T \int \phi|\nabla\omega|^2 \, dx \, ds\right)$$

(23)
$$+ \frac{K_E}{R^2}\int_0^T \int \phi^{2\rho-1}|\omega|^2 \, dx \, ds + \frac{\alpha^2 K_e}{R^4}\int_0^T \int \phi^{4\rho-3}\frac{|u|^2}{2} \, dx \, ds.$$

The space-time integrals of N_1^j and N_2^ω also enjoy formally identical bounding procedures. The spatial integrals are still split into regions depending on $|\nabla u|$. When $|\nabla u| < M$ we have the pointwise estimate $|\omega| \leq 5^{1/2}M$ and we obtain a scaled version of prior bounds. When $|\nabla u| \geq M$ we use the kinematic estimate, (14), on $|\omega|$. The same familiar argument now shows that N_1^j and N_2^ω are bounded by the same dominating quantity appearing in (22).

To summarize,

$$N := N_1^\omega + N_2^\omega + N_1^j + N_2^j \leq \frac{K_P}{\alpha} \left(\frac{1}{2} \sup_s \left(||\psi^{\frac{1}{2}}\omega||_2^2 + ||\psi^{\frac{1}{2}}j||_2^2 \right) + \int_0^T \int \phi \left(|\nabla\omega|^2 + |\nabla j|^2 \right) dx \, ds \right)$$

$$+ \frac{K_E}{R^2} \int_0^T \int \phi^{2\rho-1} \left(|\omega|^2 + |j|^2 \right) dx \, ds$$

$$+ \frac{\alpha^2 K_e}{R^4} \int_0^T \int \phi^{4\rho-3} \frac{|u|^2 + |b|^2}{2} \, dx \, ds.$$

Examining now L^ω and L^j, we note, upon integrating by parts, a cancellation occurs when we consider their sum:

$$L := L^\omega + L^j = -\int_0^T \int (j \cdot \omega)(\nabla\phi \cdot b) \, dx \, ds.$$

Although the above term is lower order and can be bounded in a more efficient way than what transpires below, we choose the less direct approach to limit problem specific dependencies of the parameter β. We again split the spatial integral to obtain on one hand that,

$$\int_0^T \int_{|\nabla u| \leq M} (j \cdot \omega)(\nabla\phi \cdot b) \, dx \, ds \leq \sqrt{5} M \frac{1}{R} \int_0^T ||\phi^{\frac{1}{2}}b||_2 ||\phi^{\frac{1}{2}}j||_2 \, dx \, ds$$

$$\leq \frac{K_E}{R^2} \int_0^T \int \phi^{2\rho-1} |j|^2 \, dx \, ds + \frac{\alpha^2 K_e}{R^4} \int_0^T \int \phi^{4\rho-3} |b|^2 \, dx \, ds.$$

On the other hand, using the kinematic decomposition of ω as a singular integral, we have,

$$\int_0^T \int_{|\nabla u| \geq M} \int_{|y| \geq R^{2/3}} \sigma(\hat{y})\omega(x+y) \frac{dy}{|y|^3} |j||\nabla\phi \cdot b| \, dx \, ds \leq \frac{1}{\alpha} \sup_s ||\psi^{\frac{1}{2}}j||_2^2$$

$$+ \frac{\alpha^2 K_e}{R^4} \int_0^T \int \phi^{4\rho-3} |b|^2 \, dx \, ds,$$

and,

$$\int_0^T \int_{|\nabla u| \geq M} \int_{|y| < R^{2/3}} \sigma(\hat{y})\omega(x+y) \frac{dy}{|y|^3} |j||\nabla\phi \cdot b| \, dx \, ds \leq \frac{1}{\alpha} \left(\int_0^T ||\phi^{\frac{1}{2}}j||_2 ||\nabla(\phi^{\frac{1}{2}}b)||_2 \, dx \, ds \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\alpha} \left(\sup_s ||\psi^{\frac{1}{2}}j||_2^2 + \int_0^T ||\phi^{\frac{1}{2}}\nabla j||_2^2 \, dx \, ds \right)$$

$$+ \frac{\alpha^2 K_e}{R^4} \int_0^T \int \phi^{4\rho-3} |b|^2 \, dx \, ds.$$

Combining the above we conclude that L is also bounded by the quantity given in (22).

To complete our discussion of bounds, observe that each of the enstrophy flux terms can also be bounded in a manner which will allow us to establish the *independent* cascades of both kinetic and magnetic enstrophies. These terms are lower order and their treatment is essentially the same as that for L. Indeed, the kinetic enstrophy flux, F^ω, ultimately satisfies the same bound as the final bound in (23) while the magnetic enstrophy flux, F^j, satisfies that from (22).

5. EXISTENCE OF THE ENSTROPHY CASCADES.

In this section Theorem 5 is proven. We work in the context of an arbitrary (K_1, K_2)-cover $\{B(x_i, R)\}_{i=1}^n$ of $B(0, R_0)$ and assume the premises of Theorem 5 hold.

First we establish lower bounds for the following quantities which are independent of i (note that x_0 was arbitrary in the previous subsection and the bounds were independent of x_0; our subscripts here indicate localization around x_i at scale R):

$$\int_0^T \Phi_{x_i,R}^\omega \, ds = \int \frac{1}{2}|\omega(x,T)|^2 \psi_i(x) \, dx + \int_0^T \int |\nabla \omega|^2 \phi_i \, dx \, ds + H_i + N_i + L_i + X_i - F_i^j,$$

$$\int_0^T \Phi_{x_i,R}^j \, ds = \int \frac{1}{2}|j(x,T)|^2 \psi_i(x) \, dx + \int_0^T \int |\nabla j|^2 \phi_i \, dx \, ds + H_i + N_i + L_i + X_i - F_i^\omega.$$

We limit our discussion to the first of these as the arguments are identical. Observe that,

$$H_i + N_i + L_i + X_i - F_i^j \leq \frac{K_P}{\alpha} \bigg(\sup_s \big(||\psi_i^{\frac{1}{2}} j||_{L^2(B(x_i,2R))} + ||\psi_i^{\frac{1}{2}} \omega||_{L^2(B(x_i,2R))} \big)$$

$$+ \int_0^T \int \phi_i \big(|\nabla j|^2 + |\nabla \omega|^2 \big) \, dx \, ds \bigg)$$

$$+ \frac{K_E}{R^2} \int_0^T \int \phi_i^{2\rho-1} \big(|\omega|^2 + |j|^2 \big) \, dx \, ds$$

$$+ \frac{\alpha^2 K_e}{R^4} \int_0^T \int \phi_i^{4\rho-3} \bigg(\frac{|b|^2}{2} + \frac{|u|^2}{2} \bigg) \, dx \, ds.$$

The properties of (K_1, K_2)-covers (see (4)) allow us, upon taking ensemble averages, to pass to a lower bound involving only integral scale quantities,

$$\bigg\langle \frac{1}{T} \int_0^T \frac{1}{R^3} \Phi_{x_i,R}^\omega \, ds \bigg\rangle_R \geq \frac{1}{K_1} P_0 - K_2 \frac{K_P}{\alpha} P_0 - \frac{K_2 K_E}{R^2} E_0 - \frac{\alpha^2 K_2 K_e}{R^4} e_0.$$

At this point we specify the value for α,

$$\alpha = 4K_P K_*^2,$$

and recall that,

$$K_* \geq \max\{(K_1 K_2)^{1/2}, 3K_2/4, K_1\}.$$

Consequently,

$$K_2 \frac{K_P}{\alpha} \leq \frac{1}{4K_1},$$

and, therefore,

$$\bigg\langle \frac{1}{T} \int_0^T \frac{1}{R^3} \Phi_{x_i,R}^\omega \, ds \bigg\rangle_R \geq \frac{3}{4K_1} P_0 - \frac{K_2 K_E}{R^2} E_0 - \frac{\alpha^2 K_2 K_e}{R^4} e_0.$$

Recalling now (A2) we have,

$$\frac{\alpha^2 K_2 K_e}{R^4} e_0 \leq \beta^4 \alpha^2 K_2 K_e P_0,$$

and,

$$\frac{K_2 K_E}{R^2} E_0 \leq \beta^2 K_2 K_E P_0.$$

Therefore, noting $0 < \beta < 1$,

$$\bigg\langle \frac{1}{T} \int_0^T \frac{1}{R^3} \Phi_{x_i,R}^\omega \, ds \bigg\rangle_R \geq \frac{3}{4K_1} P_0 - \beta^2 K_2 (K_E + \alpha^2 K_e) P_0.$$

The parameter modifying our Kraichnan-type scale, β, is now chosen so that β^2 is small enough to satisfy,

$$2\beta^2 K_1 K_2 (K_E + \alpha^2 K_e)) < 1.$$

Granted this, we obtain,

$$\left\langle \frac{1}{T} \int_0^T \frac{1}{R^3} \Phi^\omega_{x_i,R} \, ds \right\rangle_R \geq \frac{1}{4K_1} P_0 \geq \frac{1}{4K_*} P_0.$$

Establishing the upper bound is, by properties of (K_1, K_2)-covers, immediate because we chose K_* so that $4K_* \geq 3K_2$. Indeed,

$$\left\langle \frac{1}{T} \int_0^T \frac{1}{R^3} \Phi^\omega_{x_i,R} \, ds \right\rangle_R \leq K_2 P_0 + 2P_0 \leq 4K_* P_0.$$

Because the argument for $\langle \Phi^j \rangle_R$ is identical we conclude that,

$$\frac{1}{4K_*} P_0 \leq \langle \Phi^\omega \rangle_R, \langle \Phi^j \rangle_R \leq 4K_* P_0.$$

6. FLUX LOCALITY.

As mentioned previously, we can immediately deduce locality of the flux from Theorem 5. Flux locality, in the context of turbulence phenomenology, refers to the fact that the transfer takes place primarily between comparable scales. This is realized in terms of flux by the proposition that the time averaged flux at scale R is well-correlated only with the time averaged fluxes at comparable scales. While flux locality is phenomenologically accepted in the hydrodynamic case, there has been some controversy about the locality in plasma turbulence (cf. [AlDiss, AlEy10] for a locality result on the energy level, as well as a discussion and references on the topic). It is thus important to additionally examine the ratio of kinetic to magnetic enstrophy fluxes as a possible source of non-local enstrophy exchange.

Define the time-averaged local magnetic and kinetic enstrophy fluxes associated to the cover element around the point x_i as follows,

$$\Psi^\omega_{x_i,R} = \frac{1}{T} \int_0^T \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds = R^3 \Phi^\omega_{x_i,R},$$

$$\Psi^j_{x_i,R} = \frac{1}{T} \int_0^T \int \frac{1}{2} |j|^2 (u \cdot \nabla \phi_i) \, dx \, ds = R^3 \Phi^j_{x_i,R}.$$

Further, define the ensemble average over a (K_1, K_2)-cover of these time-averaged fluxes as,

$$\langle \Psi^\omega \rangle_R = \frac{1}{n} \sum_{i=1}^n \Psi^\omega_{x_i,R} = R^3 \langle \Phi^\omega \rangle_R,$$

$$\langle \Psi^j \rangle_R = \frac{1}{n} \sum_{i=1}^n \Psi^j_{x_i,R} = R^3 \langle \Phi^j \rangle_R.$$

Using the clear relationships between the spatio-temporal and ensemble averaged terms and the time and ensemble averaged terms one can use the bounds established in Theorem 5 to directly verify the following theorem (for which the proof is omitted).

Theorem 7. *Let u and b satisfy the assumptions of Theorem 5 and let R and r be two scales in the range $\sigma_0/\beta \leq r, R \leq R_0$. Then,*

$$\frac{1}{16K_*^2}\left(\frac{r}{R}\right)^3 \leq \frac{\langle\Psi^\omega\rangle_r}{\langle\Psi^\omega\rangle_R}, \frac{\langle\Psi^j\rangle_r}{\langle\Psi^j\rangle_R}, \frac{\langle\Psi^\omega\rangle_r}{\langle\Psi^j\rangle_R} \leq 16K_*^2\left(\frac{r}{R}\right)^3.$$

Remark 8. Note that along the *dyadic scale – $r = 2^k R$ –* the locality propagates *exponentially*.

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